RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. THIRD SEMESTER EXAMINATION, DECEMBER 2011

SECOND YEAR

MATHEMATICS (Honours)

Date : 16/12/2011

Time : 11am – 3 pm

Paper : III

Full Marks : 100

[Use separate answer-books for each group]

Group–A

Answer **any five** questions

1. a) Let V be a finite dimensional vector space over a field F and W be a subspace of V. Then prove that $\dim V/W = \dim V - \dim W$. 4

   b) Let V and W be two vector spaces over a field $(F,+,\bullet)$ and $T:V \to W$ be a linear transformation. Let $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be a basis of V and $\ker T = \{\theta\}$, where θ is the null vector of V. Prove that $\{T(\alpha_1), T(\alpha_2), \cdots T(\alpha_n)\}$ is a basis of Im T. 4

   c) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x,y,z) = (x+2, y+2)$ where $x,y,z \in \mathbb{R}$. Is T a linear transformation? Justify your answer. 2

2. a) Find the eigen values of a matrix A given by $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$. Find also the eigen vector corresponding to the negative eigen value of A. 3+2

   b) Define Euclidean vector space. If α, β be any two vectors in a Euclidean space prove that $|<\alpha,\beta>| \le \|\alpha\| \|\beta\|$ where $\langle \alpha, \beta \rangle$ denotes inner product of α and β and $\|\alpha\|$ denotes norm of the vector α. 2+3

3. a) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. 5

   b) If V and W be vector spaces of finite dimension over a field F and $T:V \to W$ be a linear transformation, then prove that T is invertible if and only if the matrix of T relative to any chosen pair of ordered bases of V and W is non singular. 5

4. a) A linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(0,1,1) = (1,0,1)$, $T(1,0,1) = (2,3,4)$, $T(1,1,0) = (1,2,3)$. Find the matrix of T relative to the ordered basis $\{(1,0,0),(0,1,0),(0,0,1)\}$. Deduce that T is invertible. Find the matrix of T^{-1} relative to the ordered basis $\{(1,0,0),(0,1,0),(0,0,1)\}$. 5

   b) Let V and W be two vector spaces over a field $(F,+,\bullet)$. Let $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be a basis of V and $\beta_1, \beta_2, \cdots \beta_n$ be an arbitrarily chosen set of elements in W. Prove that there exists one and only one linear mapping $T: V \to W$ such that $T(\alpha_i) = \beta_i$ for $i = 1, 2, \cdots n$. 5

5. a) Find the row space and row rank of the matrix $\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{pmatrix}$. 5

b) The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered basis {(0,1,1), (1,0,1), (1,1,0)} of $\mathbb{R}^3$ and {(1,0), (1,1)} of $\mathbb{R}^2$ is $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 2 & 0 \end{pmatrix}$. Find the explicit representation of T and the matrix of T relative to the ordered basis {(1,0,1), (1,1,0), (0,1,1)} of $\mathbb{R}^3$ and {(1,2), (1,1)} of $\mathbb{R}^2$. 3+2

6. a) State and Prove Cayley–Hamilton theorem. 1+4

b) Use Gram–Schmidt process of orthonormalisation to construct an orthonormal basis for the subspace of $\mathbb{R}^4$ generated by (1,1,0,1), (1,–2,0,0) and (1,0,–1,2). 5

7. a) Define orthogonal linear transformation. Let V be an n-dimensional real inner product space and $T: V \to V$ be a linear transformation. Prove that T is orthogonal iff $\|T(v)\| = \|v\|$ for all $v \in V$. 5

b) Let V be a finite dimensional inner product space over a field F ($\mathbb{R}$ or $\mathbb{C}$) and T be a linear operator on V. Show that there exist self adjoint operators $T_1, T_2 \in L(V, V)$ such that $T = T_1 + iT_2$. Moreover T is normal iff $T_1T_2 = T_2T_1$. 5

8. a) Let V be a finite dimensional inner product space over $F(\mathbb{R}$ or $\mathbb{C})$ and T be a linear operator on V. Suppose $\lambda \in F$ be an eigen value of T. Then show that

(i) T is unitary $\Rightarrow |\lambda| = 1$

(ii) T is Hermition $\Rightarrow \lambda$ is real

(iii) T is skew Hermition $\Rightarrow \lambda$ is purely imaginary or zero. 5

b) If α, β be any two vectors in an inner product space V over a field F ($\mathbb{R}$ or $\mathbb{C}$), then show that $\|\alpha + \beta\| \le \|\alpha\| + \|\beta\|$. 2

c) Suppose $V = \mathbb{R}^4$ and $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1 = x_2 = x_3\}$. Find the orthogonal projection of (1,1,2,0) on S. 3

Group–B

9. Answer **any three** questions : 3×5 = 15

a) If by a transformation of rotation of coordinate axes the expression $ax^2 + 2hxy + by^2$ is changed into $a'x'^2 + 2h'x'y' + b'y'^2$, then prove that $a' + b' = a + b$ and $a'b' - h'^2 = ab - h^2$. 5

b) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines of which one intersects the coordinate axes in A, B and the other intersects them in A', B', then show that the equation of the pair of lines AB' and $A'B$ is $ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0$. 5

c) Reduce the equation $3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0$ to its canonical form and determine the type of the conic represented by it. 5

d) Show that the conics $\frac{l_1}{r} = 1 - e_1 \cos\theta$ and $\frac{l_2}{r} = 1 - e_2 \cos(\theta - \alpha)$ will touch one another if $l_1^2(1-e_2^2) + l_2^2(1-e_1^2) = 2l_1l_2(1-e_1e_2\cos\alpha)$. 5

e) Prove that the locus of the poles of the normal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the curve $a^6y^2 - b^6x^2 = (a^2+b^2)^2x^2y^2$. 5

10. Answer **<u>any one</u>** question: 1x5 = 5

a) Show that the origin lies in the acute angle between the planes $x + 2y + 2z = 9$, $4x - 3y + 12z + 13 = 0$. Find the planes bisecting the angles between them, and point out which bisects the acute angle. 5

b) Show that all lines which cut the *z*-axis and the lines $\frac{x+3}{2} = \frac{y-6}{3} = \frac{z-3}{-2}$ and $\frac{x}{2} = \frac{y-6}{2} = \frac{2}{-1}$ lie on the surface

$$7(x-y+6)(x+z) = (3x-2y+21)(x+2z).$$ 5

Answer **<u>any two</u>** from Q. 11 to Q. 13:

11. a) A pair of rectangular axes are rotating in their plane with an angular velocity ω. Find the velocity and acceleration of a particle moving in the plane referred to the rotating axes. 8

b) A heavy uniform chain of length 2*l*, hangs over a small smooth fixed pulley, the length $l + c$ being at one side and $l - c$ at the other. Motion is allowed to ensue. Show that the chain will slip off the pulley in time $\sqrt{\frac{l}{g}}\log\frac{l+\sqrt{l^2-c^2}}{c}$. 7

12. a) Prove that in motion under a central force, the areal velocity is constant. 3

b) A particle describes a plane curve under the action of a central force *F* per unit mass. Prove, in usual notation, that the differential equation of the path of the particle is $\frac{h^2}{p^3}\frac{dp}{dr} = F$. 6

c) A particle is describing a circle of radius ‘a’ in such a way that its tangential acceleration is *k* times the normal acceleration, *k* being a constant. If the speed of the particle at any point be u, prove that it will return to the same point after a time $\frac{a}{ku}\left(1-\bar{e}^{2\pi k}\right)$. 6

13. a) A particle is at rest on a smooth horizontal plane which commences to turn about a straight line lying in itself with constant angular velocity ω downwards. If 'a' be the distance of the particle from the axis of rotation at zero time, show that the body will leave the plane at time t given by the equation

$$a \sinh \omega t + \frac{g}{2\omega^2} \cosh \omega t = \frac{g}{\omega^2} \cos \omega t \ .$$ 8

b) A particle of mass m, is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}\left[\lambda - \log(1+\lambda)\right]$, where V is the terminal velocity of the particle and λV is its initial vertical velocity. 7
